615–Maths Methods in Theoretical Physics

Problem Sheet 3

(1a.) Find all the singular points of the Laguerre equation,

$$x\,y'' + (1-x)\,y' + \lambda\,y = 0\,,$$

where λ is a given constant. Be sure to determine what kind of singular points arise.

(1b.) Look for analytic solutions of the Laguerre equation, as an expansion around $x = 0$ in powers of x:

$$y(x) = \sum_{n\ge 0} a_n\,x^n\,.$$

(i.e. obtain the recursion relation for the a_n.) How many linearly-independent solutions does this yield?

(1c.) Use the ratio test to determine the radius of convergence of the series. Compare this with your results in part (1a) on the locations of the singular points of the equation. For what values of λ does the series terminate?

(1d.) Show that the recursion relation is solved by

$$a_n = \frac{(-1)^n\,\Gamma(\lambda+1)}{\Gamma(\lambda+1-n)\,(n!)^2}\,a_0\,,$$

where the Gamma function $\Gamma(z)$ has the property $\Gamma(z+1) = z\,\Gamma(z)$. (When z is an integer, $\Gamma(n+1)$ is equal to $n!$.)

(2a) Show by explicit substitution into the differential equation that if $y_1(x)$ satisfies the equation $y'' + p(x)\,y' + q(x)\,y = 0$ then so does y_2, given by

$$y_2(x) = y_1(x)\int^x \frac{dt}{f(t)\,y_1^2(t)}\,, \qquad \text{where } \ f(t) = \exp\int^t p(s)\,ds\,. \qquad (1)$$

(2b) Calculate $f(t)$ for the Laguerre equation. Deduce from equation (1) that the second solution must be of the form $y_2(x) = y_1(x)\,\log x + G(x)$, where $G(x)$ is an analytic function.

(3.) Find the recursion relation for the coefficients b_n in the second solution to the Laguerre equation, which can be obtained by writing

$$y(x) = y_1(x)\,\log x + \sum_{n\ge 0} b_n\,x^n\,,$$

where y_1 is the solution obtained in question 1.

Due Tuesday 27th September in class