615–Maths Methods in Theoretical Physics

Problem Sheet 7

(**1a**) Write two quaternions A and B as ordered pairs of complex numbers, $A = (a, b)$ and $B = (c, d)$. Using the multiplication rule $A\,B = (ac - \bar{d}b, da + b\bar{c})$, show that in general multiplication is non-commutative, *i.e.* $A\,B \neq B\,A$. Show that multiplication of any three quaternions is associative; $A\,(B\,C) = (A\,B)\,C$, where $C = (e, f)$. With the conjugate defined by $\bar{A} = (\bar{a}, -b)$, show that $\bar{A}A = A\bar{A} = (a\bar{a} + b\bar{b}, 0)$, which is just the real number $a\bar{a} + b\bar{b}$.

(**1b**) Define octonions A, B and C by $A = (a, b)$, $B = (c, d)$, $C = (e, f)$, where a, b, c, d, e and f are quaternions. Using the multiplication rule $A\,B = (ac - \bar{d}b, da + b\bar{c})$, show that multiplication is non-associative.

(**1c**) The conjugate is defined by $\bar{A} = (\bar{a}, -b)$. Show that any two octonions A and B satisfy

$$\bar{A}\,(AB) = (\bar{A}A)\,B\,,$$

which is needed to establish that the octonions form a division algebra, as discussed in the lectures.

(**2**) Prove that if one defines the derivative of $f(z) = u(x, y) + \mathrm{i}\,v(x, y)$ at z by taking the limit of $(f(z + \delta z) - f(z))/\delta z$, where δz approaches zero along a line at angle ψ in the complex plane (*i.e.* take $\delta z = \epsilon\, e^{\mathrm{i}\,\psi}$ as the real constant ϵ goes to zero), then the answer is independent of ψ if the Cauchy-Riemann equations hold.

(**3.**) Show that each of the functions

$$\begin{aligned} i) \qquad & u = x^4 - 6x^2\,y^2 + y^4 \\ ii) \qquad & u = e^{x^2 - y^2}\,\cos(2x\,y) \\ iii) \qquad & u = \frac{\sin 2x}{\cosh 2y - \cos 2x} \end{aligned}$$

satisfies the Laplace equation in two dimensions. In each case, taking u to be the real part of an analytic function, use the Cauchy-Riemann equations to find $v(x, y)$, and hence find $f = u + \mathrm{i}\,v$. (In each case, express f eventually entirely in terms of z.)

Due Thursday 3rd November in class