615–Maths Methods in Theoretical Physics

Problem Sheet 8

(**1a.**) The function $f(z)$ is analytic when $|z| < R'$. Using Cauchy's integral formula, prove that if $|a| < R < R'$,

$$f(a) = \frac{1}{2\pi\,\mathrm{i}} \oint_C \frac{R^2 - |a|^2}{(z-a)(R^2 - z\,\bar{a})}\, f(z)\, dz\,,$$

where C is the circle $|z| = R$.

(**1b.**) Hence deduce Poisson's formula that, if $0 < r < R$,

$$f(re^{\mathrm{i}\,\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, f(Re^{\mathrm{i}\,\phi})\, d\phi\,.$$

(**2**) Find the Laurent expansions of the following functions around the point $z = 0$, giving each series up to and including order z^2:

$$(a) \quad \frac{1}{z^2(1-z)}\,, \qquad (b) \quad \frac{\sin z}{z^3}\,, \qquad (c) \quad \frac{1}{\tan^3 z}\,. \tag{1}$$

(**3**) A function $f(z)$ is analytic in some domain. Obtain the expansion

$$f(z) = f(a) + 2\Big[\frac{z-a}{2}\, f'\Big(\frac{z+a}{2}\Big) + \frac{(z-a)^3}{2^3\,\cdot\,3!}\, f'''\Big(\frac{z+a}{2}\Big) + \frac{(z-a)^5}{2^5\,\cdot\,5!}\, f^{(5)}\Big(\frac{z+a}{2}\Big) + \cdots\Big]\,,$$

and determine its range of validity, in terms of the domain where $f(z)$ is analytic.

(**4a.**) A function $f(z)$ is analytic within a closed contour C, and continuous on C. If $f(z)$ does not vanish anywhere within C and $|f(z)| \geq M$ for all points on C, show that $|f(z)| \geq M$ for all points *within* C. [**Hint**: Consider the function $g(z) = 1/f(z)$.]

(**4b.**) Show that this result no longer holds if $f(z)$ instead vanishes at some point or points within C, and that it is possible to have $|f(z)| > 0$ everywhere on the contour C while having $|f(z)| = 0$ at one or more points within C. Give an explicit example of an analytic function $f(z)$ with this property.

Due Thursday 10th November in class